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Markovian representations of current algebras

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Abstract. We generalize Hegerfeldt's concept of T-positivity in Euclidean random fields to non-commutative probability theory, that is, to Euclidean Fermi fields and to current algebra with possible Schwinger terms. Our axioms imply the Wightman axioms. A non-Abelian form of Markovicity is introduced, and is shown to imply T-positivity if a reflection property holds.

The investigation suggests a generalization of Nelson–Symanzik positivity, which might be valid in cases when the extension of the Schwinger functions to coinciding arguments is not expected to maintain both commutativity and positivity (or anti-commutativity and positivity).

1. Introduction and definitions

According to axiomatic quantum field theory (Wightman 1956, Streater and Wightman 1964), the Wightman functions $W(x_1, \ldots, x_n)$ of a quantum field theory possess analytic continuations to points in the forward tube

$$I = \{x \in \mathbb{C}^{4n} : \mathrm{Im}[x_i - x_{i+1}] \in V^+, i = 1, 2, \ldots n-1\}.$$

Here, $x_1 \ldots$ are complex four-vectors, and V^+ is the forward cone in $\mathbb{R}^4 : V^+ = \{x : x^0 > 0, (x^0)^2 > \boldsymbol{x}^2\}$, $x = (x^0, x^1, x^2, x^3)$. In particular, they may be continued to the *Schwinger points* $\{x \in \mathbb{C}^n : \mathrm{Re}\, x_i^0 = \mathrm{Im}\, \boldsymbol{x}_i = 0, \mathrm{Im}(x_i^0 - x_{i+1}^0) > 0\}$. Schwinger (1959) pointed out that at such points, the functions are invariant under the real Euclidean group $\mathbb{E}^\nu$; the functions W, evaluated at the Schwinger points, are called the Schwinger functions of the theory, regarded as distributions over real space and imaginary time. Symanzik (1969) had the idea that in theories based on Hamiltonians, these functions should be the moments of a generalized random field over $\mathbb{R}^4$ with certain Markov properties. He assumed that $S(x_1, \ldots x_n)$ has an extension, as a tempered distribution, to $\mathbb{R}^{4n}$, including equal-time points that are not Schwinger points.

Symanzik's ideas were given new impetus by Nelson (1973) who further developed the notion of Markov fields. This theory is related to the classical theory analysed for Gaussian fields by Wong (1969). Subsequently, Osterwalder and Schrader (1973, 1975) gave the necessary and sufficient conditions on the Schwinger functions, for them to be obtained from the Wightman functions of some relativistic quantum field. While in one sense this settles the question, it remains of interest to ask under what conditions Euclidean fields exist, or, equivalently, when can we extend the Schwinger functions to equal-time points, so as to get the moments of a positive finite measure on the space of histories. This has been discussed by Borchers and Yngvason (1976) and Challifour and Slinker (1975). The general solution in the former gives rise to a complex measure, rather than a probability measure. As a result, many of the ideas of probability theory, such as Markovicity, Feynman–Kac formula etc, lose their force. It would be interesting to find a formalism some way between the generality of the Wightman axioms (or the equivalent Osterwalder–Schrader axioms) and the much more restrictive Nelson axioms.

Hegerfeldt (1974) has proposed a new property, called T-positivity, which is more general than the Markov condition of Nelson, but which is sufficient, when combined with Nelson's other axioms for Euclidean fields, to reconstruct the Wightman theory, while maintaining a probabilistic interpretation. The T-positivity condition is similar to, but somewhat stronger than the Osterwalder–Schrader positivity condition, as is seen when it is imposed on the generating functional of the field (Hegerfeldt 1974).

The question arises whether a Euclidean probability theory, involving classical random fields, is general enough to describe all interesting Wightman fields, such as fields in the Borchers class of the free field (Wightman 1956, Streater and Wightman 1964, Borchers 1960, Epstein 1963). It is plausible that such fields obey some form of current commutation relations at coinciding points in Euclidean space. In this paper we shall allow such possibilities by introducing non-commutative random fields.

Definition 1. Let G be a Lie group with identity e and dG its Lie algebra. Let $\mathrm{Exp}: dG \to G$ be the exponential map from dG to G. Let $g(\cdot)$ be a map from $\mathbb{R}^\nu$ to G, where ν denotes the dimension of space–time. The *support* of g, written supp g, is the closure of the set $\{x: g(x) \neq e\}$. Let $\mathscr{D}(\mathbb{R}^\nu, G)$ denote the set of c^∞-maps g from $\mathbb{R}^\nu$ to G with compact support. Let $\times$ be a multiplication law on $\mathscr{D}(\mathbb{R}^\nu, G)$ making it into a group $\mathscr{G}$. We say that $\mathscr{G}$ is a *local current group* if

(1) for every f_1 and $f_2 \in \mathscr{D}(\mathbb{R}^\nu)$, and $A \in dG$, we have

$$\mathrm{Exp}(f_1(\cdot)A) \times \mathrm{Exp}(f_2(\cdot)A) = \mathrm{Exp}[(f_1(\cdot) + f_2(\cdot))A];$$

(2) for every $g_1(\cdot), g_2(\cdot) \in \mathscr{G}$ such that $\mathrm{supp}\, g_1 \cap \mathrm{supp}\, g_2 = \phi$, we have

$$g_1 \times g_2 = g_2 \times g_1.$$

That is, group elements associated with disjoint regions commute.

This definition is more general than the definition of current group given by Streater (1968, 1971) and Streater and Mathon (1971), where the group law was postulated to be pointwise multiplication: $(g_1 \times g_2)(x) = g_1(x) g_2(x)$. This pointwise definition is tantamount to the absence of Schwinger terms in the corresponding Lie algebra. Hermann (1970) has shown how to define a group whose Lie algebra contains Schwinger terms of any desired form. Hermann's groups are local current groups according to definition 1.

Another example of a local current group is the Schwartz space $\mathscr{D}(\mathbb{R}^\nu)$ itself, being $\mathscr{D}(\mathbb{R}^\nu, \mathbb{R})$ with pointwise addition taking the role of $\times$.

So far, we have not put any topology on the current group. It does not seem possible to do this in an interesting way if we insist that our group becomes locally compact. Instead, we shall impose a mild continuity condition on the representations to be considered.

2. Markovian and T-positive representations

If $\mathscr{H}$ is a Hilbert space, we shall denote by Aut $\mathscr{H}$ the group of unitary operators on $\mathscr{H}$.

Let $(\mathscr{D}(\mathbb{R}^\nu, G), \times)$ be a local current group and suppose $f \in \mathscr{D}(\mathbb{R}^\nu)$ and $A \in dG$ are given. Then the *one-parameter subgroup through* fA is the map $\mathbb{R} \to \mathscr{D}(\mathbb{R}^\nu, G)$ given by $\lambda \to \mathrm{Exp}(\lambda f(\cdot)A)$.

Definition 2. A cyclic representation of the current group $\mathscr{G} = (\mathscr{D}(\mathbb{R}^\nu, G), \times)$ on a Hilbert space $\mathscr{H}$, with cyclic vector Ω, is a group homomorphism U from $\mathscr{G}$ to Aut $\mathscr{H}$, such that $U(g)$ is continuous along every one-parameter subgroup, and the vectors $\{U(g(\cdot))\Omega: g(\cdot) \in \mathscr{G}\}$ are total in $\mathscr{H}$.

As usual, a cyclic representation is uniquely determined by the expectation functional

$$F(g(\cdot)) = \langle \Omega, U(g(\cdot))\Omega \rangle.$$

Any generalized random field in the sense of Gel'fand and Vilenkin (1964) provides a representation of $\mathscr{D}(\mathbb{R}^\nu)$, and any representation of the canonical commutation relations in Segal's (1963) form provides a representation of $\mathscr{D}(\mathbb{R}^\nu, N)$, where N is the Heisenberg group and the multiplication is pointwise.

An analysis of all representations known as factorizable was undertaken by Streater (1969) and more completely, by Araki (1970). Parthasarathy and Schmidt (1972a) study factorizable projective representations (see Guichardet 1972 and Parthasarathy and Schmidt 1972b for a general account). Goldin and Sharp (1970) and Grodnik and Sharp (1970) give examples of representations of groups without pointwise multiplication, and a systematic way to allow for Schwinger terms is to be found in Parthasarathy and Schmidt (1976). For further work, see Vershik *et al* (1973, 1974, 1975) and Albeverio and Høegh-Krohn (1976). None of these constructions is useful for relativistic quantum field theory, even if interpreted in Euclidean space–time rather than as equal-time commutations relations in $\mathbb{R}^3$. This is because the factorizability condition leads to fields with independent values at different points, which is too strong a requirement. We now replace this condition by a more appropriate condition, T-positivity, following Hegerfeldt (1974) closely.

Definition 3. We say a cyclic representation $(U, \Omega, \mathscr{H})$ of a local current group $(\mathscr{D}(\mathbb{R}^\nu, G), x)$ is Euclidean covariant if there exists a continuous representation $(a, R) \to T(a, R)$ of the full Euclidean group on $\mathscr{H}$, and a representation $R \to S(R)$ of $O(\nu)$ on dG, regarded as a real vector space, such that $T(a, R)\Omega = \Omega$ and

$$T(a, R)U(\mathrm{Exp}(f(\cdot)A))T^{-1}(a, R) = U(\mathrm{Exp}\, f_{a,R}(\cdot)S(R)A)$$

for all $(a, R) \in \mathbb{E}^\nu$, $f \in \mathscr{D}(\mathbb{R}^\nu)$, $A \in dG$. Here, $f_{a,R}(x) = f(R^{-1}(x-a))$, $x \in \mathbb{R}^\nu$.

If $(U, \Omega, \mathscr{H})$ is a Euclidean covariant representation of a local current group, we shall denote by T the unitary operator $T(0, \theta)$ where $\theta \in O(\nu)$ is time-reversal

$$\theta(x_1, \ldots, x_\nu) = (x_1, x_2, \ldots x_{\nu-1}, -x_\nu).$$

T_t will denote $T((0, 0, \ldots, t), 1)$.

Let $\mathbb{R}^\nu_\pm = \{x \in \mathbb{R}^\nu: \pm x^\nu > 0\}$ and let $E_\pm$ be the projection onto the subspace $\mathscr{H}_\pm$, which is that generated by $\{U(g(\cdot))\Omega$: supp $g \in \mathbb{R}^\nu_\pm\}$.

Definition 4. With this notation, the representation is said to be T-positive if

$$E_+TE_+ \geqslant 0.$$

As in Hegerfeldt (1974), one can then decompose $\mathscr{H}_+$ into $\mathscr{H}_0$ on which E_+TE_+ is strictly positive, and the null space $\mathscr{k}_N$ of E_+TE_+. Let E_0 be the orthogonal projection onto $\mathscr{H}_0$ and T_0 the restriction of E_0TE_0 to $\mathscr{H}_0$; then T_0^{-1} exists as a densely defined operator. Analogues of lemmas 2.1, 2.2 and proposition 2.1 of Hegerfeldt (1974) can then be formulated and proved as done by Hegerfeldt leading to the following proposition and theorem.

Proposition 2.1. Let $(U, \Omega, \mathscr{H})$ be a T-positive cyclic Euclidean covariant representation of a local current group, and let $\mathscr{H}_0$, T_0, E_0 and T_t be as above. Let $P_t = T_0^{1/2}E_0T_tT_0^{-1/2}$ on $T_0^{1/2}\mathscr{H}_0$. Then $\{P_t, t \geq 0\}$ can be extended to a continuous self-adjoint contraction semi-group on $\mathscr{H}_0$.

Hegerfeldt (1974) shows that T-positivity is a generalization of a weak form of Markovicity, and the reflection property of Nelson. The same considerations apply here. If $(U, \Omega, \mathscr{H})$ is a cyclic representation of a current group, covariant but not necessarily T-positive, we may say that it is Markovian relative to the plane $x^\nu = 0$ if E_+E_- is a projection, E_0. Let $\mathscr{H}_0 = E_0\mathscr{H}$. This represents states localized at $x^\nu = 0$. Let us say such a representation obeys the reflection property if $T = 1$ on $\mathscr{H}_0$.

Theorem. Let $\mathscr{G} = (\mathscr{D}(\mathbb{R}^\nu, G), x)$ be a local current group, and $(U, \Omega, \mathscr{H})$ a cyclic representation of $\mathscr{G}$, covariant under time-translation and time-reflection. Then the representation is Markovian and satisfies the reflection property if and only if E_+TE_+ is a projection.

3. The axiom (A′) and the reconstruction theorem

Let $(U, \Omega, \mathscr{H})$ be a cyclic representation of a current group $\mathscr{G} = (\mathscr{D}(\mathbb{R}^\nu, G), \times)$, which is covariant under translations. Let $K = (\Sigma P_j^2)^{1/2}$, where P_j, $j = 1, 2, \ldots \nu$, is the generator of space-translations in $\mathbb{R}^\nu$. Let $\mathscr{k}^k$ be the associated scale, $-\infty \leq k \leq +\infty$. Let $A(f)$ denote the self-adjoint generator of $\lambda \to U(\text{Exp}\, \lambda f(\cdot)A)$, $\lambda \in \mathbb{R}$, $A \in dG$, $f \in \mathscr{D}(\mathbb{R}^\nu)$.

Assumption (A'). $\langle u, A(f)v \rangle$ is defined and is separately continuous on $\mathscr{k}^\infty \times \mathscr{D}(\mathbb{R}^\nu) \times \mathscr{k}^\infty$.

Let $\mathscr{S}_<(\mathbb{R}^{\nu n}) \subseteq \mathscr{S}(\mathbb{R}^{\nu n})$ denote the set of functions such that for any derivative $f^{(\alpha)}$, $f^{(\alpha)}(x_1, \ldots x_n) = 0$ unless $x_1^\nu < x_2^\nu < \ldots < x_n^\nu$.

Theorem. Let $\mathscr{G} = (\mathscr{D}(\mathbb{R}^\nu, G), \times)$ be a local current group, and let $(U, \Omega, \mathscr{H})$ be a cyclic representation, covariant under the full Euclidean group, with unique translation-invariant vector Ω (up to a multiple). Assume T-positivity and assumption A′. Then for $A_1, \ldots A_n \in dG$, $\langle \Omega, A_1(f_1) \ldots A_n(f_n)\Omega \rangle$ exist as tempered distributions on $\mathscr{S}_<(\mathbb{R}^{\nu_n})$, and are the Schwinger functions of a unique Wightman theory.

The proof of this theorem follows, lemma by lemma, the proof of theorem (4.1) of Hegerfeldt (1974). At one point, the author remarks that the commutativity of the fields is crucial; but detailed investigation reveals that it is only necessary that the Euclidean fields commute at different times.

Just as in Hegerfeldt (1974) one identifies $\mathscr{H}_0$ as the physical Hilbert space of the Wightman theory, and if $P_t = e^{-Ht}$, H is the Hamiltonian.

So far, we have considered unitary cyclic representations of groups. The Wightman axioms, and the Osterwalder–Schrader axioms, are formulated in terms of the fields $A(f)$ rather than the unitary operators $e^{iA(f)}$. The formulation in terms of groups is tighter, since it ensures the self-adjointness of $A(f)$ while avoiding any assumptions about its domain. Nevertheless, it is useful to reformulate our axioms in terms of the fields. Since we make no specific assumptions about the singularity at coincident points, one can drop the assumptions that the field has values in a Lie algebra, replacing it by a general vector space, and one may also allow fermions.

Axioms

(1) We are given a Hilbert space $\mathscr{H}$, carrying a representation $T(a, R)$ of the full Euclidean group $\mathbb{E}$, with a unique invariant vector Ω.
(2) Let $K = (\Sigma P_j^2)^{1/2}$, where $T(a, 1) = e^{iaP}$ and let $\mathscr{k}^k$, $-\infty \leq k \leq \infty$, be the corresponding scale.

We are given fields $A_j(x)$ such that $\langle u, A_j(f)v\rangle$ is a continuous map from $\mathscr{k}^\infty \times \mathscr{S}(\mathbb{R}^\nu) \times \mathscr{k}^\infty$ to $\mathbb{C}$, and

$$T(a, R)A_j(f)T^{-1}(a, R) = \sum_k S_{jk}(R)A_k(f_{a,R})$$

for $(a, R) \in \mathbb{E}$, $f \in \mathscr{S}(\mathbb{R}^\nu)$.
(3) If $\operatorname{supp} f \cap \operatorname{supp} g = \varnothing$, $A_j(f)A_k(g) = \pm A_k(g)A_j(f)$.
(4) Let T denote time-reversal, and let $\mathscr{H}^+$ be the span of $\{A_{j_1}(f_1) \ldots A_{j_n}(f_n)\Omega;\ n = 0, 1, \ldots f_i \in \mathscr{S}(\mathbb{R}_+^\nu), j_1, \ldots \in (0, 1, \ldots)\}$. Let E_+ be the projection onto $\mathscr{H}_+$. Then $E_+TE_+ \geq 0$ holds.

Such a system will be called a *T-positive quantum field.*

The reconstruction theorem:

if $(A, \mathscr{H}, \Omega)$ is a T-positive quantum field, then

$$\langle \Omega, A_{j_1}(f_1) \ldots A_{j_n}(f_n)\Omega\rangle, \qquad \text{defined on } \mathscr{S}_<(\mathbb{R}^\nu),$$

are the Schwinger functions of a unique Wightman field. The proof is as in Hegerfeldt (1974).

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